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A CONTRIBUTION TO THE THEORY OF LINEAR HOMOGENEOUS GEOMETRIC DIFFERENCE EQUATIONS

(q-DIFFERENCE EQUATIONS)

BY

FOLKE RYDE

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A CONTRIBUTION TO THE THEORY OF LINEAR HOMOGENEOUS GEOMETRIC DIFFERENCE EQUATIONS
(q-DIFFERENCE EQUATIONS)

BY

FOLKE RYDE

BY DUE PERMISSION OF THE PHILOSOPHICAL FACULTY OF LUND
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BERLINGSKA BOKTRYCKERIET

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Introduction.

A functional relation of the following type

$$\sum_{i=0}^{i=n} A_i(x) \, . \, u(xq^i) = 0, \tag{1}$$

where $A_0(x)$, $A_1(x)$, $\ldots A_n(x)$ are known functions of the complex variable x and where q is a real or imaginary constant while $u(x)$ is the function to be determined, may be termed a linear homogeneous geometric difference equation (or q-difference equation) of the n-th order. The aim of the present paper is to bring the theory of these equations, at least on some points, to the same degree of perfection as the theories of the corresponding differential equations and arithmetic (ordinary) difference equations as they have been built up the former principally by Fuchs and Frobenius (1) * the latter by Nörlund (1). This equation (1) is a particular case of a general class of functional relations considered by Grévy (1), following ideas set forth by Kœnigs (1), (2), (3). The equation (1) has then been treated in detail by Carmichael (1), who employs the method of successive approximations. He also indicates a possibility of solving the equation by aid of power series, which idea has then been worked out in a particular case by Mason (1). In a still more particular case power series have been used already by Heine (1) and Thomae (1), (2) and later on by Jackson (4) and Smith (1). It is, however, possible to replace the power series by another class of series, which may be termed geometric factorial series. These series appear in many respects to be more natural to the problem in question, just as arithmetic (ordinary)

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* Such a number placed in a parenthesis close by a name refers to the list of literature p. 43.

factorial series have proved to be with regard to arithmetic difference equations as shown by Nörlund (*loc. cit.*). A similar series has been used already by Cauchy (1) in a particular case. By aid of these series one is immediately led to a system of linearly independent solutions, which are *uniform*, analytic functions of the variable x as soon as the coefficients $A_i(x) (i = 0, 1, 2, \ldots n)$ of the difference equation are uniform functions of x satisfying certain conditions. This seems to be the principal advantage of the method. From these solutions one may easily pass to those which have been obtained by Grévy and Carmichael.

In order to give a survey of the subject treated in this paper, I will here indicate the contents of its particular sections. In § 1 the main properties of geometric factorial series are summarized. In § 2 the proof of the existence of a system of solutions of the difference equation in question is given. In § 3 these solutions are examined with regard to their asymptotic character and a proof of their linear independence is given. In § 4 an examination of the analytic character of the solutions is undertaken, and the solutions given by Grévy and Carmichael are deduced. In § 5 some applications to particular equations are made. The list of literature p. 43 contains only works quoted in the paper. Of these I will especially mention the already cited works of Frobenius and Nörlund, of which I have made frequent use.

In this context I desire to express my sincere thanks to Professor N. E. Nörlund, who has drawn my attention to the problem in question and whose kind interest and valuable suggestions have been of great importance to me in preparing this paper.

§ 1.

By a geometric factorial series in x we mean a series of the following type

$$\alpha_0 + \sum_{s=1}^{s=\infty} \frac{\alpha_s}{(1-x)\left(1-\frac{x}{q}\right)\left(1-\frac{x}{q^2}\right)\cdots\left(1-\frac{x}{q^{s-1}}\right)}, \tag{1}$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_s, \ldots$ are given real or imaginary constants or functions of q, where q is an arbitrary real or imaginary constant, of which we suppose that $0 < |q| < 1$; if not, the series presents quite different properties. This series is a particular case of more general classes of series that have sometimes been considered, thus by Frobenius (2), Pincherle (1), Tonelli (1) and Appell (1), (2), (3), (4). A series of this particular type (1) has been used already by Jacobi (1) in his researches in elliptic functions. A detailed treatment of it has been commenced by Jackson (1), (2), (3). As our method to a large extent was to be based upon these series, it was necessary to examine them in detail. This can be done by means of quite elementary considerations in close connection with the theory of power series *. In this context I will give only a summary of their main properties. For this purpose, however, I employ the integral-theory of Cauchy, which, though less elementary in principle, proves to be more convenient in application.

At first I refer to the following easily shown property of geometric factorials $(1-x)(1-xq)(1-xq^2)\ldots(1-xq^s)$ (cf. Frobenius, *loc. cit.* (2)): It is always possible for every positive integer s and for every value of the variable x, situated in a given finite region of the x-plane, which does not enclose any point out of the following set $x = q^{-t}$ $(t = 0, 1, 2, \ldots s)$, to determine two positive constants m and M, which are independent of s and x, in such a way that

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* I intend to publish these researches elsewhere.

$$0 < m < |(1-x)(1-xq)\ldots(1-xq^s)| < M. \tag{2}$$

This is a consequence of the uniform convergence of the infinite product $(1-x)(1-xq)(1-xq^2)\ldots$ for points x situated in a finite region. By aid of these inequalities one immediately finds that it is generally possible to determine such a circle about the origin as a centre that the series (1) converges uniformly without it, and thus here represents a holomorphic function of x. If we suppose that

$$\alpha_s = a_s\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q^3}\right)\ldots\left(1-\frac{1}{q^{s-1}}\right)$$

we can obtain such a circle by choosing its radius $>$ 1 and R, where

$$R = \overline{\lim_{s\to\infty}} \sqrt[s]{|a_s|}, \tag{3}$$

according to the well-known theorem of Cauchy and Hadamard on power series. The latter quantity R may be termed the convergence-radius of the series (1).

Hence it is a necessary condition for the possibility of developing a function in geometric factorial series of type (1) that the function is holomorphic in the neighbourhood of infinity. This condition, however, is also sufficient. We suppose that $G(x)$ is a function holomorphic for $|x| > r$. Then we have, according to Cauchy's integral theorem (cf. Osgood, *Lehrbuch der Funktionentheorie*, 2. Aufl., Bd. 1, p. 327),

$$G(x) = G(\infty) + \frac{1}{2\pi i}\cdot\int\limits_{(\Gamma)}\frac{G(z)}{z-x}dz,$$

where the integration is extended in positive direction with respect to infinity along a circle Γ whose radius ϱ is chosen in such a way that $|x| > \varrho > r$ and whose centre is the origin. By aid of the easily verified identity

$$\frac{1}{z-x}=\sum_{s=0}^{s=n}\frac{q^{-s}\,.\,(1-z)\left(1-\frac{z}{q}\right)\ldots\left(1-\frac{z}{q^{s-1}}\right)}{(1-x)\left(1-\frac{x}{q}\right)\left(1-\frac{x}{q^2}\right)\ldots\left(1-\frac{x}{q^s}\right)}+$$

$$+\frac{1}{z-x}\cdot\frac{(1-z)\left(1-\frac{z}{q}\right)\ldots\left(1-\frac{z}{q^n}\right)}{(1-x)\left(1-\frac{x}{q}\right)\ldots\left(1-\frac{x}{q^n}\right)} \tag{4}$$

this gives

$$G(x)=G(\infty)+\frac{1}{2\pi i}\cdot\int\limits_{(\Gamma)}G(z)\,.\sum_{s=0}^{s=n}\frac{q^{-s}\,.\,(1-z)\left(1-\frac{z}{q}\right)\ldots\left(1-\frac{z}{q^{s-1}}\right)}{(1-x)\left(1-\frac{x}{q}\right)\ldots\left(1-\frac{x}{q^s}\right)}dz+R_n,$$

where

$$R_n=\frac{1}{2\pi i}\cdot\int\limits_{(\Gamma)}\frac{G(z)}{z-x}\cdot\frac{(1-z)\left(1-\frac{z}{q}\right)\ldots\left(1-\frac{z}{q^n}\right)}{(1-x)\left(1-\frac{x}{q}\right)\ldots\left(1-\frac{x}{q^n}\right)}dz.$$

R_n tends by indefinitely increasing n in limit to zero as

$$|R_n|<\frac{M\varrho}{|x|-\varrho}\cdot\left|\frac{\varrho}{x}\right|^{n+1}\cdot\frac{K}{k},$$

where M is the maximum value of $|G(z)|$ along the circle Γ and K and k denote two finite positive constants, $k\neq 0$; further is according to our supposition $|x|>\varrho$. Thus we have the following development

$$G(x)=G(\infty)+\sum_{s=0}^{s=\infty}\frac{g_s}{(1-x)\left(1-\frac{x}{q}\right)\ldots\left(1-\frac{x}{q^s}\right)}, \tag{5}$$

valid for $|x|>r$ [*], where

$$g_s=\frac{q^{-s}}{2\pi i}\cdot\int\limits_{(\Gamma)}G(z)\,.\,(1-z)\left(1-\frac{z}{q}\right)\ldots\left(1-\frac{z}{q^{s-1}}\right)dz. \tag{5'}$$

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[*] We always exclude those points at which the denominators of the terms of the series vanish.

Hence if a function admits of a development in series of type (1), this development is possible in only one way for a given value of q.

The expression (5′) enables us to give an upper bound to the coefficients g_s, of which we will make use later on. We have

$$|g_s| < \frac{1}{2\pi} \cdot \frac{1}{|q^s|} \cdot \left| \int_{(\Gamma)} G(z) \,.\, (1-z)\left(1-\frac{z}{q}\right) \ldots \left(1-\frac{z}{q^{s-1}}\right) dz \right|$$

$$\therefore |g_s| < \frac{M\varrho}{\varkappa^s} \cdot (1+\varrho)\left(1+\frac{\varrho}{\varkappa}\right) \ldots \left(1+\frac{\varrho}{\varkappa^{s-1}}\right)$$

$$\therefore |g_s| < M \,.\, (1+\varrho)\left(1+\frac{\varrho}{\varkappa}\right) \ldots \left(1+\frac{\varrho}{\varkappa^s}\right), \tag{6}$$

where M as above denotes the maximum value of $|G(z)|$ along a circle of radius ϱ about the origin as a centre and where $\varkappa = |q|$.

A somewhat more general development of the same kind may be obtained by using the following identity instead of (4)

$$\frac{1}{z-x} = t \cdot \sum_{s=0}^{s=n} \frac{q^{-s} \,.\, (1-zt)\left(1-\frac{zt}{q}\right) \ldots \left(1-\frac{zt}{q^{s-1}}\right)}{(1-xt)\left(1-\frac{xt}{q}\right) \ldots \left(1-\frac{xt}{q^s}\right)} +$$

$$+ \frac{1}{z-x} \cdot \frac{(1-zt)\left(1-\frac{zt}{q}\right) \ldots \left(1-\frac{zt}{q^n}\right)}{(1-xt)\left(1-\frac{xt}{q}\right) \ldots \left(1-\frac{xt}{q^n}\right)}, \tag{7}$$

where t is an arbitrary real or imaginary constant. This gives

$$G(x) = G(\infty) + \sum_{s=0}^{s=\infty} \frac{\bar{g}_s}{(1-xt)\left(1-\frac{xt}{q}\right) \ldots \left(1-\frac{xt}{q^s}\right)}, \tag{8}$$

valid for $|x| > r$, where

$$\bar{g}_s = \frac{1}{2\pi i} \cdot \frac{t}{q^s} \cdot \int_{(\Gamma)} G(z) \cdot (1-zt)\left(1-\frac{zt}{q}\right) \ldots \left(1-\frac{zt}{q^{s-1}}\right) dz. \tag{8′}$$

Of this expression for $\bar{g}_s$ we obtain as above

$$|\bar{g}_s| < M.(1+\varrho\tau)\left(1+\frac{\varrho\tau}{\varkappa}\right)\ldots\left(1+\frac{\varrho\tau}{\varkappa^s}\right), \tag{9}$$

where M and $\varkappa$ have the same meaning as before and where $\tau = |t|$. A comparison between (5′) and (8′) shows that the coefficients $\bar{g}_s$ may be expressed as linear functions of $g_0, g_1, g_2, \ldots g_s$ with coefficients that are polynomials in t. Thus we have performed a transformation of the series (5) into the series (8), which is analogous to the corresponding transformation of arithmetic factorial series, treated in detail by Nörlund (2).

§ 2.

It is evident that by a simple substitution the linear homogeneous geometric difference equation

$$\sum_{i=0}^{i=n} A_i(x) \,.\, u(xq^i) = 0, \tag{1}$$

where $|q|$ is supposed to be >1, may be transformed into an analogous equation, where $|q|$ is <1. Hence it is in this respect no real limitation to suppose that $|q|$ is <1, as we do. If on the contrary $|q|=1$ the problem assumes quite another character.

To begin with we perform a formal transformation of the equation (1) by introducing geometric differences, which we define thus (compare Jackson (4))

$$Qu(x) = Q^1u(x) = \frac{u(xq)-u(x)}{xq-x} \text{ and } Q^n u(x) = Q(Q^{n-1}u(x)). \tag{2}$$

$$(n = 2, 3, 4, \ldots.)$$

Then we have as may be easily verified by induction

$$(q-1)^\nu \cdot q^{\frac{1}{2}\nu(\nu-1)} \cdot x^\nu \cdot Q^\nu u(x) =$$

$$= \sum_{i=0}^{i=\nu} (-1)^{\nu-i} \cdot \left[{\nu \atop i}\right] \cdot q^{\frac{1}{2}(\nu-i)(\nu-i-1)} \cdot u(xq^i), \tag{3}$$

the first term of the sum on the right-hand side being $(-1)^{\nu} \cdot q^{\frac{1}{2}\nu(\nu-1)} \cdot u(x)$ and the last term $u(xq^{\nu})$. Further we have denoted

$$\left[{\nu \atop i}\right] = \frac{(1-q^{\nu-i+1})(1-q^{\nu-i+2})\dots(1-q^{\nu})}{(1-q)(1-q^2)\dots(1-q^i)}.$$

Conversely

$$u(xq^{\nu}) = \sum_{i=0}^{i=\nu} \left[{\nu \atop i}\right] \cdot (q-1)^i \cdot q^{\frac{1}{2}i(i-1)} \cdot x^i \cdot Q^i u(x), \tag{4}$$

the first term of the sum being $u(x)$. Hence the given equation (1) may be transformed into the following form

$$\sum_{h=0}^{h=n} B_h(x) \cdot x^h \cdot Q^h u(x) = 0,$$

where $B_h(x)$ is a linear combination of the functions $A_h(x)$, $A_{h+1}(x)$, ... $A_n(x)$ with coefficients that are independent of the variable x. We suppose that the coefficients $A_i(x)$ and thus also $B_i(x)$ are holomorphic in the neighbourhood of infinity. Hence we may transform the difference equation (1) into the following form

$$\sum_{h=0}^{h=n} G_h(x) \cdot (1-xq)(1-xq^2)\dots(1-xq^h) \cdot Q^h u(x) = 0, \tag{5}$$

the first term being $G_0(x) \cdot u(x)$, where the coefficients $G_h(x)$ are holomorphic in the neighbourhood of infinity and thus may be supposed to be given as geometric factorial series of type (1) of § 1

$$G_h(x) = a_0^{(h)} + \sum_{s=0}^{s=\infty} \frac{a_{s+1}^{(h)}}{(1-x)\left(1-\frac{x}{q}\right)\dots\left(1-\frac{x}{q^s}\right)}. \quad (h=0, 1, 2, \dots n) \tag{5'}$$

This series is supposed to have the convergence-radius R_h.

It is interesting to observe that this difference equation (5), at least if the functions $G_h(x)$ are independent of q, when q approaches unity tends to a differential equation, which after a simple transformation gives an equation of the type that was treated by Frobenius in his already quoted classical investigation. (Frobenius (1).)

Now we start from an equation of this type supposing that $a_0^{(n)} \neq 0$, and hence we may without loss of generality suppose that $G_n(x)$ is independent of the variable x and equal to $(q-1)^n : q^{\frac{1}{2}n(n+1)}$.

By $\Omega[q, x]$, or, on account of simplicity, $\Omega[x]$ we denote the following function, whose importance to the analysis was set forth by Heine (1) (compare Ashton (1) and for a generalization of the product functions Appell (5))

$$\Omega[x] = \frac{(1-q)(1-q^2)(1-q^3)\ldots}{(1-xq)(1-xq^2)(1-xq^3)\ldots}. \quad (0 < |q| < 1) \tag{6}$$

$\Omega[x]$ is a meromorphic function of x with poles of the first order at the points $x = q^{-s}$ $(s = 1, 2, 3, \ldots)$, for which infinity is an essentially singular point. It is evident that $\Omega[x]$ satisfies the following relation

$$\Omega[x] = (1-x)\left(1-\frac{x}{q}\right)\left(1-\frac{x}{q^2}\right)\ldots\left(1-\frac{x}{q^{s-1}}\right)\cdot\Omega\left[\frac{x}{q^s}\right].$$

Hence a geometric factorial series of the type considered in § 1 may be written thus

$$\sum_{s=0}^{s=\infty} \alpha_s \cdot \frac{\Omega\left[\frac{x}{q^s}\right]}{\Omega[x]}.$$

We will now attempt to satisfy the difference equation in question (5) by the following series

$$u(x) = \sum_{\nu=0}^{\nu=\infty} g_\nu \cdot \frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]} = \frac{\Omega[xr]}{\Omega[x]} \cdot \sum_{\nu=0}^{\nu=\infty} \frac{g_\nu}{(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{\nu-1}}\right)}, \tag{7}$$

where the constant r and the coefficients g_ν are to be determined. We have as may be easily verified by induction

$$Q^i\left(\frac{\Omega[xr]}{\Omega[x]}\right) = \frac{q^{\frac{1}{2}i(i+1)}}{(q-1)^i} \cdot \frac{(1-r)\left(1-\frac{r}{q}\right)\ldots\left(1-\frac{r}{q^{i-1}}\right)}{(1-xq)(1-xq^2)\ldots(1-xq^i)} \cdot \frac{\Omega[xr]}{\Omega[x]}.$$

Let the left-hand member of the difference equation (5) be denoted $P(u(x))$ and suppose $(h = 0, 1, 2, \ldots n)$

$\overline{G}_h(x)=q^{\frac{1}{2}h(h+1)}\cdot(q-1)^{-h}\cdot G_h(x)$ and $\bar{a}_s^{(h)}=q^{\frac{1}{2}h(h+1)}\cdot(q-1)^{-h}\cdot a_s^{(h)}$, (8)
where $G_h(x)$ is the series given by (5′). Then we have

$$P(u(x))=\sum_{\nu=0}^{\nu=\infty} g_\nu\cdot P\left(\frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]}\right)$$

and

$$P\left(\frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]}\right)=\frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]}\cdot\sum_{i=0}^{i=n}\overline{G}_i(x)\cdot\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\cdots\left(1-\frac{r}{q^{\nu+i-1}}\right).$$

Hence we get

$$P(u(x))=\sum_{\nu=0}^{\nu=\infty} g_\nu\cdot\frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]}\cdot\left(\sum_{i=0}^{i=n}\overline{G}_i(x)\cdot\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\cdots\left(1-\frac{r}{q^{\nu+i-1}}\right)\right).$$

Let R denote the greatest of the quantities R_0, R_1, … R_n and 1. Then all the functions $\overline{G}_h(x)(h=0,1,2,\ldots n)$ are holomorphic for $|x|>R$ and admit, according to the transformation given in § 1, of the following development in geometric factorial series

$$\overline{G}_h(x)=\sum_{s=0}^{s=\infty}\frac{b_s^{(h)}}{\left(1-\frac{xr}{q^\nu}\right)\left(1-\frac{xr}{q^{\nu+1}}\right)\cdots\left(1-\frac{xr}{q^{\nu+s-1}}\right)},\tag{9}$$

valid for $|x|>R$ at least. The first constant term does not change by this transformation. Hence $b_0^{(h)}=\bar{a}_0^{(h)}$, while for $s\geqq 1$ $b_s^{(h)}$ may be expressed as a linear function of $\bar{a}_1^{(h)}$, $\bar{a}_2^{(h)}$, … $\bar{a}_s^{(h)}$ with coefficients that are polynomials in $rq^{-\nu}$.

If we then suppose

$$\varphi(x,t)=\sum_{i=0}^{i=n}\overline{G}_i(x)\cdot(1-t)\left(1-\frac{t}{q}\right)\cdots\left(1-\frac{t}{q^{i-1}}\right)$$

we have

$$\varphi\left(x,\frac{r}{q^\nu}\right)=\sum_{s=0}^{s=\infty}\frac{\sum_{i=0}^{i=n}\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\cdots\left(1-\frac{r}{q^{\nu+i-1}}\right)\cdot b_s^{(i)}}{\left(1-\frac{xr}{q^\nu}\right)\left(1-\frac{xr}{q^{\nu+1}}\right)\cdots\left(1-\frac{xr}{q^{\nu+s-1}}\right)}.$$

Further we denote

$$\varphi_0(r) = \sum_{i=0}^{i=n} (1-r)\left(1-\frac{r}{q}\right)\cdots\left(1-\frac{r}{q^{i-1}}\right)\cdot \bar{a}_0^{(i)} \tag{10}$$

and more generally

$$\varphi_s\left(\frac{r}{q^\nu}\right) = \sum_{i=0}^{i=n} \left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\cdots\left(1-\frac{r}{q^{\nu+i-1}}\right)\cdot b_s^{(i)}. \tag{10'}$$

$\varphi_s\left(\frac{r}{q^\nu}\right)$ may then be expressed linearly in $\bar{a}_i^{(\varkappa-1)}$ $(\varkappa = 1, 2, 3, \ldots n$ and $i = 1, 2, 3, \ldots s)$ with coefficients that are polynomials in $rq^{-\nu}$. Now we have

$$\varphi\left(x, \frac{r}{q^\nu}\right) = \sum_{s=0}^{s=\infty} \frac{\varphi_s\left(\frac{r}{q^\nu}\right)}{\left(1-\frac{xr}{q^\nu}\right)\left(1-\frac{xr}{q^{\nu+1}}\right)\cdots\left(1-\frac{xr}{q^{\nu+s-1}}\right)}. \tag{11}$$

The substitution of the series (7) in our difference equation (5) gives therefore

$$\begin{aligned}
P(u(x)) &= \sum_{\nu=0}^{\nu=\infty} g_\nu \cdot \frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]} \cdot \varphi\left(x, \frac{r}{q^\nu}\right) \\
&= \sum_{\nu=0}^{\nu=\infty} g_\nu \cdot \frac{\Omega\left[\frac{xr}{q^\nu}\right]}{\Omega[x]} \cdot \left(\sum_{s=0}^{s=\infty} \frac{\varphi_s\left(\frac{r}{q^\nu}\right)}{\left(1-\frac{xr}{q^\nu}\right)\left(1-\frac{xr}{q^{\nu+1}}\right)\cdots\left(1-\frac{xr}{q^{\nu+s-1}}\right)}\right) \\
&= \sum_{\nu=0}^{\nu=\infty} g_\nu \cdot \left(\sum_{s=0}^{s=\infty} \varphi_s\left(\frac{r}{q^\nu}\right) \cdot \frac{\Omega\left[\frac{xr}{q^{\nu+s}}\right]}{\Omega[x]}\right) \\
&= \sum_{\mu=0}^{\mu=\infty} \frac{\Omega\left[\frac{xr}{q^\mu}\right]}{\Omega[x]} \cdot \left(g_0 \cdot \varphi_\mu(r) + g_1 \cdot \varphi_{\mu-1}\left(\frac{r}{q}\right) + \cdots + g_\mu \cdot \varphi_0\left(\frac{r}{q^\mu}\right)\right).
\end{aligned}$$

We thus obtain a formal solution of the difference equation (5) by choosing the quantities $r, g_0, g_1, g_2, \ldots$ so as to satisfy the following system of equations

$$\left.\begin{array}{l}
g_0 \cdot \varphi_0(r) = 0, \\
g_0 \cdot \varphi_1(r) + g_1 \cdot \varphi_0\left(\frac{r}{q}\right) = 0, \\
g_0 \cdot \varphi_2(r) + g_1 \cdot \varphi_1\left(\frac{r}{q}\right) + g_2 \cdot \varphi_0\left(\frac{r}{q^2}\right) = 0, \\
- - - - - - - - - - - - - - - - - - \\
g_0 \cdot \varphi_\nu(r) + g_1 \cdot \varphi_{\nu-1}\left(\frac{r}{q}\right) + \cdots + g_{\nu-1} \cdot \varphi_1\left(\frac{r}{q^{\nu-1}}\right) + g_\nu \cdot \varphi_0\left(\frac{r}{q^\nu}\right) = 0, \\
- -
\end{array}\right\} \qquad (12)$$

Now we suppose that $g_0 \neq 0$ so that the first term of the series (7) is $g_0 \cdot \frac{\Omega[xr]}{\Omega[x]}$. Then the first of the equations (12) gives the following relation to determine r

$$\varphi_0(r) = 0, \qquad (13)$$

i. e., according to (10),

$$\sum_{i=0}^{i=n} (1-r)\left(1-\frac{r}{q}\right)\cdots\left(1-\frac{r}{q^{i-1}}\right) \cdot \bar{a}_0^{(i)} = 0. \qquad (13')$$

This rational integral equation in r, whose degree is n, since $\bar{a}_0^{(n)} = 1$, we term the *indicial equation* of the geometric difference equation (5). We suppose that this equation has no vanishing roots.

Now we proceed in the same manner as Frobenius (1) in case of linear differential equations or Nörlund (1) in case of linear arithmetic difference equations. We exclude for a moment the first of the equations (12) and consider r as a variable, which, however, is admitted to assume only such values as are situated in the neighbourhood of the particular roots of the indicial equation. This region of variation for r, which thus may be composed of several distinct regions, we denote D. As the roots of the indicial equation are supposed to be finite and not zero, it is possible to determine two positive constants l and L so that for every value of r belonging to D

$$0 < l < |r| < L. \qquad (14)$$

The remaining equations (12) then leave the coefficient g_0 undetermined. If we suppose, however, that g_0 is an arbitrary function of r, say $g_0(r)$, $g_1, g_2, g_3, \ldots$ may be determined successively as functions of r. Thus we obtain

$$g_\nu(r)=\frac{(-1)^\nu\cdot g_0(r)}{\varphi_0\left(\frac{r}{q}\right)\cdot\varphi_0\left(\frac{r}{q^2}\right)\cdots\varphi_0\left(\frac{r}{q^\nu}\right)}\cdot\begin{vmatrix}\varphi_1(r), & \varphi_0\left(\frac{r}{q}\right), & 0 & ,\ldots & 0\\ \varphi_2(r), & \varphi_1\left(\frac{r}{q}\right), & \varphi_0\left(\frac{r}{q^2}\right), & \ldots & 0\\ - & - & - & - & -\\ \varphi_\nu(r), & \varphi_{\nu-1}\left(\frac{r}{q}\right), & \varphi_{\nu-2}\left(\frac{r}{q^2}\right), & \ldots & \varphi_1\left(\frac{r}{q^{\nu-1}}\right)\end{vmatrix}.$$

Hence it follows that

$$g_\nu(r)=\frac{g_0(r)\cdot h_\nu(r)}{\varphi_0\left(\frac{r}{q}\right)\cdot\varphi_0\left(\frac{r}{q^2}\right)\cdots\varphi_0\left(\frac{r}{q^\nu}\right)},\tag{15}$$

where $h_\nu(r)$ denotes a polynomial in r. Those points in the r-plane at which the denominator of this fractional expression vanishes are situated on equiangular spirals traced through the points that represent the roots of the indicial equation (13). (When q is real, these spirals become straight lines.) It is obvious that one may always limit the region D in such a way that the denominator in question does not vanish at any other points within D than at those which represent the roots of the indicial equation. We will now classify these roots in the following manner: all the roots that differ from each other only by factors consisting of powers of q with integer exponents are grouped together. Such a group may be

$$r_i,\ r_i\cdot q^{\alpha_1^{(i)}},\ r_i\cdot q^{\alpha_2^{(i)}},\ldots r_i\cdot q^{\alpha_s^{(i)}},$$

where $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, ... $\alpha_s^{(i)}$ denote positive integers, which are supposed to be arranged in such a way that $\alpha_m^{(i)}<\alpha_n^{(i)}$, if $m<n$. r_i is then the individual of the group that has the greatest absolute value. All the individuals of a group are situated on one and the same of the above-mentioned equiangular spirals. Let ε be the greatest of the numbers $\alpha_s^{(i)}$ for all the groups. We then suppose

$$g_0(r)=\varphi_0\left(\frac{r}{q}\right)\cdot\varphi_0\left(\frac{r}{q^2}\right)\cdots\varphi_0\left(\frac{r}{q^\varepsilon}\right)\cdot C(r),\tag{16}$$

where $C(r)$ is an arbitrary function of r, of which we suppose that it is bounded and does not vanish in D. Then we are able to determine the functions $g_\nu(r)$ out of the system (12) and these functions are finite for every r belonging to the region D.

Thus we have got a formal solution of the non-homogeneous geometric difference equation

$$P(u(x)) = \frac{\Omega[xr]}{\Omega[x]} \cdot g_0(r) \cdot \varphi_0(r) \tag{17}$$

in form of the following series

$$u(x) = u(x, r) = \sum_{\nu=0}^{\nu=\infty} g_\nu(r) \cdot \frac{\Omega\left[\dfrac{xr}{q^\nu}\right]}{\Omega[x]}$$

$$= \frac{\Omega[xr]}{\Omega[x]} \cdot \sum_{\nu=0}^{\nu=\infty} \frac{g_\nu(r)}{(1 - xr)\left(1 - \dfrac{xr}{q}\right) \dots \left(1 - \dfrac{xr}{q^{\nu-1}}\right)}.$$

It remains for us to prove that this series converges. We can do that by following a way due to Frobenius. (*Loc. cit.* (1), p. 218.) As $\varphi_0\left(\frac{r}{q^{\nu+1}}\right) \neq 0$ for $\nu \geqq \varepsilon$ the $(\nu+2)$-th of the equations (12) gives

$$g_{\nu+1} = -\frac{1}{\varphi_0\left(\dfrac{r}{q^{\nu+1}}\right)} \cdot \left\{ g_0 \cdot \varphi_{\nu+1}(r) + g_1 \cdot \varphi_\nu\left(\frac{r}{q}\right) + \dots + g_\nu \cdot \varphi_1\left(\frac{r}{q^\nu}\right) \right\}.$$

Hence we have

$$|g_{\nu+1}| \leqq \frac{1}{\left|\varphi_0\left(\dfrac{r}{q^{\nu+1}}\right)\right|} \cdot \left\{ |g_0| \cdot |\varphi_{\nu+1}(r)| + |g_1| \cdot \left|\varphi_\nu\left(\frac{r}{q}\right)\right| + \dots + |g_\nu| \cdot \left|\varphi_1\left(\frac{r}{q^\nu}\right)\right| \right\}.$$

Now $\varphi\left(x, \frac{r}{q^\nu}\right)$ is a holomorphic function of x for $|x| > R$, for which we have already given the development (11) in geometric factorial series. Then we have according to (9) of § 1

$$\left|\varphi_s\left(\frac{r}{q^\nu}\right)\right| < M_\nu \cdot \left(1 + \frac{K\varrho}{\varkappa^\nu}\right)\left(1 + \frac{K\varrho}{\varkappa^{\nu+1}}\right) \dots \left(1 + \frac{K\varrho}{\varkappa^{\nu+s-1}}\right),$$

where M_ν denotes the maximum value of $\left|\varphi\left(x, \frac{r}{q^\nu}\right)\right|$ along a circle whose centre is the origin and whose radius is K, which quantity is chosen greater than R, while $\varrho = |r|$ and $\varkappa = |q|$. Hence

$$\begin{aligned}|g_{\nu+1}| < \frac{1}{\left|\varphi_0\left(\frac{r}{q^{\nu+1}}\right)\right|}\Bigg\{ & |g_\nu|\cdot M_\nu\cdot\left(1+\frac{K\varrho}{\varkappa^\nu}\right)+ \\ & +|g_{\nu-1}|\cdot M_{\nu-1}\cdot\left(1+\frac{K\varrho}{\varkappa^\nu}\right)\left(1+\frac{K\varrho}{\varkappa^{\nu-1}}\right)+ \\ & +|g_{\nu-2}|\cdot M_{\nu-2}\cdot\left(1+\frac{K\varrho}{\varkappa^\nu}\right)\left(1+\frac{K\varrho}{\varkappa^{\nu-1}}\right)\left(1+\frac{K\varrho}{\varkappa^{\nu-2}}\right)+ \\ & - - - - - - - - - - - - - - \\ & +|g_0|\cdot M_0\cdot\left(1+\frac{K\varrho}{\varkappa^\nu}\right)\left(1+\frac{K\varrho}{\varkappa^{\nu-1}}\right)\ldots(1+K\varrho)\Bigg\}.\end{aligned}$$

We denote the right-hand member of this inequality $A_{\nu+1}$. Then we have

$$A_{\nu+1}=\frac{1+\frac{K\varrho}{\varkappa^\nu}}{\left|\varphi_0\left(\frac{r}{q^{\nu+1}}\right)\right|}\left\{|g_\nu|\cdot M_\nu+\left|\varphi_0\left(\frac{r}{q^\nu}\right)\right|\cdot A_\nu\right\},$$

while according to our notation $|g_\nu| < A_\nu$, and thus

$$A_{\nu+1} < A_\nu\cdot\left(1+\frac{K\varrho}{\varkappa^\nu}\right)\cdot\left\{\frac{M_\nu}{\left|\varphi_0\left(\frac{r}{q^{\nu+1}}\right)\right|}+\left|\frac{\varphi_0\left(\frac{r}{q^\nu}\right)}{\varphi_0\left(\frac{r}{q^{\nu+1}}\right)}\right|\right\}.$$

We have defined

$$\varphi\left(x,\frac{r}{q^\nu}\right)=\sum_{i=0}^{i=n}\overline{G}_i(x)\cdot\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\ldots\left(1-\frac{r}{q^{\nu+i-1}}\right).$$

Let m_i $(i=0,\ 1,\ 2,\ldots n)$ denote the maximum value of $|\overline{G}_i(x)|$ along the circle of radius K about the origin as a centre. Then

$$\begin{aligned}M_\nu \leqq \sum_{i=0}^{i=n} m_i\cdot\left(1+\frac{\varrho}{\varkappa^\nu}\right)\left(1+\frac{\varrho}{\varkappa^{\nu+1}}\right)\ldots\left(1+\frac{\varrho}{\varkappa^{\nu+i-1}}\right)=\varkappa^{-\frac{1}{2}n(n-1)}\cdot\Bigg\{\left(\frac{\varrho}{\varkappa^\nu}\right)^n+ \\ +\psi_{n-1}\left(\frac{\varrho}{\varkappa^\nu}\right)\Bigg\},\end{aligned}$$

since $m_n=1$. Here $\psi_{n-1}\left(\frac{\varrho}{\varkappa^\nu}\right)$ denotes a polynomial in $\frac{\varrho}{\varkappa^\nu}$ of a degree not higher than $n-1$ with positive coefficients. Further is

$$|\varphi_0(r)| \leqq \varkappa^{-\frac{1}{2}n(n-1)} \cdot \{\varrho^n + \chi_{n-1}(\varrho)\},$$

since $\bar{a}_0^{(n)} = 1$. Here $\chi_{n-1}(\varrho)$ also denotes a polynomial in ϱ of a degree not higher than $n-1$ with positive coefficients. On the other hand we have

$$\begin{aligned} |\varphi_0(r)| &\geqq \varkappa^{-\frac{1}{2}n(n-1)} \cdot \varrho^n - |\varphi_0(r) - q^{-\frac{1}{2}n(n-1)} \cdot (-r)^n| \\ &\geqq \varkappa^{-\frac{1}{2}n(n-1)} \cdot \{\varrho^n - \chi_{n-1}(\varrho)\} > 0, \end{aligned}$$

at least if ϱ is chosen sufficiently large. Then we have for sufficiently large ν

$$\begin{aligned} &\frac{M_\nu}{|\varphi_0(rq^{-\nu-1})|} + \left|\frac{\varphi_0(rq^{-\nu})}{\varphi_0(rq^{-\nu-1})}\right| \\ &\leqq \frac{(\varrho\varkappa^{-\nu})^n + \psi_{n-1}(\varrho\varkappa^{-\nu})}{(\varrho\varkappa^{-\nu-1})^n - \chi_{n-1}(\varrho\varkappa^{-\nu-1})} + \frac{(\varrho\varkappa^{-\nu})^n + \chi_{n-1}(\varrho\varkappa^{-\nu})}{(\varrho\varkappa^{-\nu-1})^n - \chi_{n-1}(\varrho\varkappa^{-\nu-1})} \\ &< \frac{(L\varkappa^{-\nu-1})^n + \psi_{n-1}(L\varkappa^{-\nu-1})}{(l\varkappa^{-\nu-1})^n - \chi_{n-1}(L\varkappa^{-\nu-1})} + \frac{(L\varkappa^{-\nu-1})^n + \chi_{n-1}(L\varkappa^{-\nu-1})}{(l\varkappa^{-\nu-1})^n - \chi_{n-1}(L\varkappa^{-\nu-1})}. \end{aligned}$$

As this latter expression, which does not depend on r, converges to a finite limit when ν tends to infinity, it is possible to determine an integer N, greater than ε, and a positive constant S so that for every $\nu \geqq N$ and for every r belonging to the region D

$$\frac{M_\nu}{|\varphi_0(rq^{-\nu-1})|} + \left|\frac{\varphi_0(rq^{-\nu})}{\varphi_0(rq^{-\nu-1})}\right| < S.$$

Hence it follows that for $\nu \geqq N$

$$A_{\nu+1} < A_\nu \cdot \left(1 + \frac{K\varrho}{\varkappa^\nu}\right) \cdot S$$

and thus

$$A_{N+\nu} < A_N \cdot \left(1 + \frac{K\varrho}{\varkappa^N}\right)\left(1 + \frac{K\varrho}{\varkappa^{N+1}}\right) \cdots \left(1 + \frac{K\varrho}{\varkappa^{N+\nu-1}}\right) \cdot S^\nu.$$

Then we have for $\nu = 0, 1, 2, \ldots$

$$\left|\frac{g_{N+\nu}(r)}{(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{N+\nu-1}}\right)}\right| < \frac{A_{N+\nu}}{\left|(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{N+\nu-1}}\right)\right|}$$

$$< A_N \cdot \frac{\left(1+\frac{K\varrho}{\varkappa^N}\right)\left(1+\frac{K\varrho}{\varkappa^{N+1}}\right)\ldots\left(1+\frac{K\varrho}{\varkappa^{N+\nu-1}}\right)}{\left|(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{N+\nu-1}}\right)\right|} \cdot S^\nu < \bar{A}_N \cdot \frac{M}{m} \cdot \frac{(KS)^\nu}{|x|^\nu},$$

where M and m have the same meaning as in (2) of § 1 and thus are independent of x and r while $\bar{A}_N$ denotes an upper bound to A_N within D. Here $|x|$ is supposed to be greater than l^{-1} and thus $|xr| > 1$. Let $\bar{R}$ be chosen greater than l^{-1} and KS. Then we have for every x for which $|x| > \bar{R}$ and for every r within D

$$\left|\frac{g_{N+\nu}(r)}{(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{N+\nu-1}}\right)}\right| < \bar{A}_N \cdot \frac{M}{m} \cdot \left(\frac{KS}{\bar{R}}\right)^\nu,$$

where the right-hand member depends neither on x nor on r. Hence it follows that the series

$$v(x, r) = \sum_{\nu=0}^{\nu=\infty} \frac{g_\nu(r)}{(1-xr)\left(1-\frac{xr}{q}\right)\ldots\left(1-\frac{xr}{q^{\nu-1}}\right)}$$

converges absolutely and uniformly at least for every x situated without the circle $|x| = \bar{R}$ and for every r belonging to D, since the series of positive terms

$$\sum_{\nu=0}^{\nu=\infty} \bar{A}_N \cdot \frac{M}{m} \cdot \left(\frac{KS}{\bar{R}}\right)^\nu$$

converges. According to a theorem of Weierstrass $v(x, r)$ then represents an analytic function of x and r, holomorphic in the above-mentioned regions and may therefore be differentiated with respect to r an arbitrary number of times and the series thus obtained converge on the same conditions. Thus

$$w(x)=w(x,r)=\frac{\partial^{\varkappa}}{\partial r^{\varkappa}}u(x,\,r)=\sum_{i=0}^{i=\varkappa}\binom{\varkappa}{i}\cdot\frac{\partial^{i}}{\partial r^{i}}\frac{\Omega[xr]}{\Omega[x]}\cdot\frac{\partial^{\varkappa-i}}{\partial r^{\varkappa-i}}v(x,\,r) \tag{18}$$

is a solution of the non-homogeneous difference equation

$$P(w(x))=\frac{\partial^{\varkappa}}{\partial r^{\varkappa}}\left\{\frac{\Omega[xr]}{\Omega[x]}\cdot g_0(r)\cdot\varphi_0(r)\right\}. \tag{19}$$

Now we are also able to treat the case when the indicial equation has equal roots. We have above p. 15 made a classification of the roots. Let r_0, r_1, $\ldots$ r_λ, $\ldots$ r_ν, $\ldots$ r_π be one of those groups of roots in which the individuals have been arranged in such a way that $r_\lambda : r_\nu$ is a power of q whose exponent is a negative integer if $\lambda<\nu$. It may happen that this quotient is $=1$ and thus equal roots exist. Now we suppose that of the roots belonging to the group in question

α are equal, namely $r_0=r_1=r_2=\ldots=r_{\alpha-1}$,
$\beta-\alpha$ » » » $r_\alpha=r_{\alpha+1}=r_{\alpha+2}=\ldots=r_{\beta-1}$,
$\gamma-\beta$ » » » $r_\beta=r_{\beta+1}=r_{\beta+2}=\ldots=r_{\gamma-1}$,
— — — — — — — — — — — — — — — — — —
$\pi-\omega+1$ » » » $r_\omega=r_{\omega+1}=r_{\omega+2}=\ldots=r_\pi$.

Then according to (16)

$g_0(r)$ does not vanish for $r=r_0=r_1=r_2=\ldots=r_{\alpha-1}$,
$g_0(r)$ vanishes of order α for $r=r_\alpha=r_{\alpha+1}=r_{\alpha+2}=\ldots=r_{\beta-1}$,
$g_0(r)$ » » » β » $r=r_\beta=r_{\beta+1}=r_{\beta+2}=\ldots=r_{\gamma-1}$,
— —
$g_0(r)$ » » » ω » $r=r_\omega=r_{\omega+1}=r_{\omega+2}=\ldots=r_\pi$.

Hence

$g_0(r)\cdot\varphi_0(r)$ vanishes of order α for $r=r_0=r_1=r_2=\ldots=r_{\alpha-1}$,
$g_0(r)\cdot\varphi_0(r)$ » » » β » $r=r_\alpha=r_{\alpha+1}=r_{\alpha+2}=\ldots=r_{\beta-1}$,
$g_0(r)\cdot\varphi_0(r)$ » » » γ » $r=r_\beta=r_{\beta+1}=r_{\beta+2}=\ldots=r_{\gamma-1}$,
— —
$g_0(r)\cdot\varphi_0(r)$ » » » $\pi+1$ » $r=r_\omega=r_{\omega+1}=r_{\omega+2}=\ldots=r_\pi$.

Therefore

$$\left[\frac{\partial^{\nu}}{\partial r^{\nu}}g_0(r)\cdot\varphi_0(r)\right]_{r=r_\varkappa}\text{ vanishes for }\nu=0,1,2,\ldots\varkappa.$$

The right-hand side of the equation (19) then vanishes identically for $r = r_\varkappa$\*, and thus we obtain that

$$\overline{w}(x) = w(x, r_\varkappa) = \left[\frac{\partial^\varkappa}{\partial r^\varkappa} u(x, r)\right]_{r = r_\varkappa} \quad (\varkappa = 0, 1, 2, \dots \pi)$$

is a solution of the geometric difference equation

$$P(\overline{w}(x)) = 0.$$

If we denote the roots of the indicial equation classified in groups as above $r_0^{(s)}, r_1^{(s)}, r_2^{(s)}, \dots r_{\pi_s}^{(s)}$ $(s = 0, 1, 2, \dots k)$, where some of the groups may contain only one root and where two or more roots of a group may be equal, we thus obtain the following system of solutions of the difference equation (5)

$$\left.\begin{aligned} u_0^{(s)}(x) &= u(x, r_0^{(s)}), \\ u_1^{(s)}(x) &= \left[\frac{\partial}{\partial r} u(x, r)\right]_{r = r_1^{(s)}}, \\ u_2^{(s)}(x) &= \left[\frac{\partial^2}{\partial r^2} u(x, r)\right]_{r = r_2^{(s)}}, \\ &\text{— — — — — — — —} \\ u_{\pi_s}^{(s)}(x) &= \left[\frac{\partial^{\pi_s}}{\partial r^{\pi_s}} u(x, r)\right]_{r = r_{\pi_s}^{(s)}}. \end{aligned}\right\} (s = 0, 1, 2, \dots k). \tag{20}$$

We can easily calculate the derivatives $\frac{\partial^i}{\partial r^i}\left(\frac{\Omega[xr]}{\Omega[x]}\right)$ that appear in our solutions. We have, according to (6),

$$\frac{\Omega[xr]}{\Omega[x]} = \frac{(1 - xq)(1 - xq^2)(1 - xq^3)\dots}{(1 - xrq)(1 - xrq^2)(1 - xrq^3)\dots},$$

where the infinite products are uniformly convergent if x and r are limited to finite regions. Hence it follows that

$$\frac{\partial}{\partial r}\left(\frac{\Omega[xr]}{\Omega[x]}\right) = \frac{\Omega[xr]}{\Omega[x]} \cdot \left(\frac{xq}{1 - xrq} + \frac{xq^2}{1 - xrq^2} + \dots\right).$$

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* We exclude points coinciding with the poles of the function $\Omega[xr_\varkappa]$.

Thus we are led to consider the following functions (cf. Appell, *loc. cit.* (5), p. 95 and Rausenberger (1), p. 459)

$$\Phi^{(n)}[q; x, r] = \Phi^{(n)}[x, r] = \sum_{\nu=1}^{\nu=\infty}\left(\frac{xq^{\nu}}{1 - xrq^{\nu}}\right)^{n}. \tag{21}$$

It is obvious that $\Phi^{(n)}[x, r]$ is a holomorphic function of r and x for small values of $|x|$ and $|r|$ if $|q| < 1$, and that

$$(n-1)!\,.\,\Phi^{(n)}[x, r] = \frac{\partial^{n-1}}{\partial r^{n-1}}\Phi^{(1)}[x, r]. \tag{22}$$

For fixed values of r and q ($|q| < 1$) the series $\Phi^{(n)}[x, r]$ converges uniformly for all points x situated within an arbitrary finite region that does not enclose any point $x = 1 : rq^{\nu}$ ($\nu = 1, 2, 3, \ldots$). (In particular for $r = 0$, $\Phi^{(n)}[x, r]$ reduces to $\dfrac{q^{n}}{1 - q^{n}}\,.\,x^{n}$, which case we exclude in the sequel.) If $n = 1$ and $r = 1$ we obtain the Heinean Φ-function. It is worthy of remark that this function $\Phi^{(1)}[x, 1]$ for $x = 1$ reduces to the well-known Lambertian series in q. In this context the close connection that exists between this function and the logarithmic function may also be mentioned (compare the note p. 41).

Now we have

$$\frac{\partial}{\partial r}\frac{\Omega[xr]}{\Omega[x]} = \frac{\Omega[xr]}{\Omega[x]}\cdot\Phi^{(1)}[x, r],$$

$$\frac{\partial^{2}}{\partial r^{2}}\frac{\Omega[xr]}{\Omega[x]} = \frac{\Omega[xr]}{\Omega[x]}\cdot 1!\,.\,\Phi^{(2)}[x, r] + \frac{\partial}{\partial r}\frac{\Omega[xr]}{\Omega[x]}\cdot\Phi^{(1)}[x, r],$$

— — — — — — — — — — — — — — —

$$\frac{\partial^{i}}{\partial r^{i}}\frac{\Omega[xr]}{\Omega[x]} = \frac{\Omega[xr]}{\Omega[x]}\cdot(i-1)!\,.\,\Phi^{(i)}[x, r] + \ldots + \frac{\partial^{s}}{\partial r^{s}}\frac{\Omega[xr]}{\Omega[x]}\cdot\frac{(i-1)!}{s!}\cdot\Phi^{(i-s)}[x, r] +$$
$$+ \ldots + \frac{\partial^{i-1}}{\partial r^{i-1}}\frac{\Omega[xr]}{\Omega[x]}\cdot\Phi^{(1)}[x, r].$$

This system gives resolved with respect to $\dfrac{\partial^{i}}{\partial r^{i}}\dfrac{\Omega[xr]}{\Omega[x]}$

$$\frac{\partial^{i}}{\partial r^{i}}\frac{\Omega[xr]}{\Omega[x]} = \frac{\Omega[xr]}{\Omega[x]}\cdot\varphi^{(i)}(x, r), \tag{23}$$

where $\varphi^{(i)}(x, r)$ is a rational integral function of $\Phi^{(1)}[x, r]$, $\Phi^{(2)}[x, r], \ldots$ $\Phi^{(i)}[x, r]$ of degree i, since the term of highest degree is $(\Phi^{(1)}[x, r])^i$.

Note: We have hitherto supposed that the coefficients $A_i(x)$ of the difference equation (1) are holomorphic in the neighbourhood of infinity. It is obvious that one may by simple substitutions to this case transform also that when these coefficients are holomorphic in the neighbourhood of the origin. If the coefficients are holomorphic in the neighbourhood of infinity as well as of the origin, we thus obtain two systems of solutions of the difference equation in question. The relation that exists between them has been investigated under certain hypotheses by Carmichael, *loc. cit.* (1). (Compare Birkhoff (1), p. 559.)

§ 3.

The solutions obtained in the preceding paragraph generally have an essential singularity at infinity on account of the nature of the function $\Omega[x]$. Hence one cannot expect to get a simple exhaustive description of the behaviour of these functions in the neighbourhood of infinity. Valuable information in this respect may, however, be obtained in the following way.

In the x-plane we define a certain region R, which may be termed the *fundamental region.* Consider two circles C and C^1 with centre at the origin and passing through the points $x = 1$ and $x = q$ respectively. Then the region R shall consist of every point within the circular ring enclosed by C and C^1 and of every point belonging to C. Let x be an arbitrary point in the plane. Then every point $\frac{x}{q^s}$ $(s = 1, 2, 3, \ldots)$ is said to be *externally congruent* to x and every point $\frac{x}{q^s}$ $(s = -1, -2, -3, \ldots)$ is said to be *internally congruent* to x. (Cf. Carmichael, *loc. cit.* (1), p. 158.) In the same manner there correspond to every circle Γ congruent circles, whose points are congruent to those of Γ. Now it is obvious that to every point x in the plane there corresponds one and only one positive, negative or vanishing integer s, such that the point $\overline{x} = xq^s$ belongs to the fundamental region R. This integer that depends upon x

as well as upon q we term the *height* of the point x (cf. Kœnigs, *loc. cit.* (2), p. 7) and denote $h[x, q]$ or $h[x]$. Hence

$$x = \frac{\overline{x}}{q^{h[x]}},$$

where $\overline{x}$ belongs to R.

Now consider the Heinean function

$$O[x] = (1 - xq)(1 - xq^2)(1 - xq^3)\ldots.$$

Here we have

$$O\left[\frac{\overline{x}}{q^s}\right] = \frac{(-\overline{x})^s}{q^{\frac{1}{2}s(s-1)}} \cdot \left(1 - \frac{1}{\overline{x}}\right)\left(1 - \frac{q}{\overline{x}}\right)\ldots\left(1 - \frac{q^{s-1}}{\overline{x}}\right) . O[\overline{x}]$$

and hence

$$O[x] = (-x)^{h[x]} . q^{\frac{1}{2}h[x].(h[x]+1)} . \left(1 - \frac{1}{\overline{x}}\right)\left(1 - \frac{q}{\overline{x}}\right)\ldots\left(1 - \frac{q^{s-1}}{\overline{x}}\right) . O[\overline{x}]. \quad (1)$$

Let γ be a circle of arbitrarily small radius ε and centre at the point $x = 1$ and construct all the circles congruent to γ. Then we infer that if x tends to infinity in quite an arbitrary manner with the only restriction never to enter any of these circles

$$\left| O[x] . (-x)^{-h[x]} . q^{-\frac{1}{2}h[x].(h[x]+1)} \right| \quad (2)$$

remains between limits that are finite and do not vanish. For then we have

$$0 < \overline{m} < \left| \left(1 - \frac{1}{\overline{x}}\right)\left(1 - \frac{q}{\overline{x}}\right)\ldots\left(1 - \frac{q^{s-1}}{\overline{x}}\right) . O[\overline{x}] \right| < \overline{M}$$

independently of s and $\overline{x}$. (Compare p. 6.) This result seems worthy of remark on account of its simplicity. (The asymptotic character of the function $O[x]$ has been investigated by several writers, thus by Mellin, Barnes, Hardy and Littlewood *.)

Now consider the function $\dfrac{\Omega[x\varrho]}{\Omega[xr]} = \dfrac{O[xr]}{O[x\varrho]}$. As above we obtain

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* For detailed references compare Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 5, p. 395 (1907).

$$\frac{\Omega[x\varrho]}{\Omega\,[xr]}=\frac{r^{h[x]}}{\varrho^{h[x]}}\cdot\frac{\left(1-\dfrac{1}{\overline{x}r}\right)\left(1-\dfrac{q}{\overline{x}r}\right)\ldots\left(1-\dfrac{q^{s-1}}{\overline{x}r}\right).\,O[\overline{x}r]}{\left(1-\dfrac{1}{\overline{x}\varrho}\right)\left(1-\dfrac{q}{\overline{x}\varrho}\right)\ldots\left(1-\dfrac{q^{s-1}}{\overline{x}\varrho}\right).\,O[\overline{x}\varrho]}$$

and hence that

$$\left|\frac{\varrho^{h[x]}}{r^{h[x]}}\cdot\frac{\Omega[x\varrho]}{\Omega[xr]}\right| \tag{3}$$

remains between limits that are finite and do not vanish if x tends to infinity in quite an arbitrary manner with the only restriction never to enter circles congruent to two circles of arbitrarily small radii and centres at the points $x=r^{-1}$ and $x=\varrho^{-1}$.

Moreover by the same limit-passage

$$\lim_{x\to\infty}\frac{\Omega[x\varrho]}{\Omega[xr]}=0 \tag{4}$$

if $|\varrho|>|r|>0$.

Now we pass to the functions $\Phi^{(n)}[x,r]$ defined in the preceding paragraph. $\Phi^{(n)}[x,r]$ satisfies as may be easily verified the following difference equation

$$\Phi^{(n)}\left[\frac{x}{q},r\right]=\left(\frac{x}{1-xr}\right)^{n}+\Phi^{(n)}[x,r],$$

or

$$\Phi^{(n)}\left[\frac{x}{q},r\right]=(-r)^{-n}.\left(1+\frac{\dfrac{1}{xr}}{1-\dfrac{1}{xr}}\right)^{n}+\Phi^{(n)}[x,r],$$

or

$$\Phi^{(n)}\left[\frac{x}{q},r\right]=(-r)^{-n}+(-r)^{-n}.\sum_{\nu=1}^{\nu=n}\binom{n}{\nu}\cdot\left(\frac{\dfrac{1}{xr}}{1-\dfrac{1}{xr}}\right)^{\nu}+\Phi^{(n)}[x,r].$$

Hence we have

$$\Phi^{(n)}\left[\frac{\overline{x}}{q^{s}},r\right]=(-r)^{-n}\cdot s+(-r)^{-n}.\sum_{\nu=1}^{\nu=n}\binom{n}{\nu}\cdot\left\{\sum_{\lambda=0}^{\lambda=s-1}\left(\frac{\dfrac{q^{\lambda}}{\overline{x}r}}{1-\dfrac{q^{\lambda}}{\overline{x}r}}\right)^{\nu}\right\}+\Phi^{(n)}[\overline{x},r]$$

and therefore

$$\Phi^{(n)}[x, r] = (-r)^{-n} . h[x] + (-r)^{-n} . \sum_{\nu=1}^{\nu=n} \binom{n}{\nu} . \left\{ \sum_{\lambda=0}^{\lambda=s-1} \left(\frac{\frac{q^\lambda}{\overline{x} r}}{1 - \frac{q^\lambda}{\overline{x} r}} \right)^\nu \right\} + \Phi^{(n)}[\overline{x}, r].$$

Thus we infer that

$$\lim_{x \to \infty} \frac{\Phi^{(n)}[x, r]}{h[x]} = (-r)^{-n} \tag{5}$$

and hence

$$\lim_{x \to \infty} \frac{1}{\Phi^{(n)}[x, r]} = 0 \tag{6}$$

when x tends to infinity in quite an arbitrary manner with the only restriction never to enter circles congruent to a circle of arbitrarily small radius and centre at the point $x = r^{-1}$. For then we have since for every integer s $|1 - \overline{x} r q^s| > \delta$, where δ denotes a positive constant,

$$|\Phi^{(n)}[\overline{x}, r]| < \text{a positive constant}$$

independently of $\overline{x}$ and further

$$\left| \sum_{\lambda=0}^{\lambda=s-1} \left(\frac{\frac{q^\lambda}{\overline{x} r}}{1 - \frac{q^\lambda}{\overline{x} r}} \right)^\nu \right| < \sum_{\lambda=0}^{\lambda=\infty} \left| \frac{\frac{q^\lambda}{\overline{x} r}}{1 - \frac{q^\lambda}{\overline{x} r}} \right|^\nu < \text{a positive constant}$$

independently of s and of $\overline{x}$.

Moreover we conclude that for arbitrary r, ϱ, n and ν

$$\lim_{x \to \infty} \frac{\Phi^{(n)}[x, r]}{\Phi^{(\nu)}[x, \varrho]} = \lim_{x \to \infty} \frac{\Phi^{(n)}[x, r]}{h[x]} \cdot \frac{h[x]}{\Phi^{(\nu)}[x, \varrho]} = \frac{(-\varrho)^\nu}{(-r)^n} \tag{7}$$

excluding by the limit-passage circles congruent to two circles of arbitrarily small radii and centres at the points $x = r^{-1}$ and $x = \varrho^{-1}$.

Finally I will point at the connection that exists between the method used above and the ideas set forth by Gaston Julia (1), (2) in his interesting, recently published researches on entire and meromorphic functions in connection with Picard's well-known theorem.

Now we are able to investigate the asymptotic character of the solutions obtained in the preceding paragraph. By aid of (23) of § 2 we infer that the dominant term of $\frac{\partial^i}{\partial r^i}\frac{\Omega[xr]}{\Omega[x]}$ for great absolute values of the variable x is $\frac{\Omega[xr]}{\Omega[x]}\cdot(\Phi^{(1)}[x, r])^i$. Here as in the sequel we exclude certain congruent circles defined as above. Then we conclude by aid of (18) of § 2 that for great absolute values of the variable x the dominant term of the solutions

$$u_k(x)=\left[\frac{\partial^k}{\partial r^k}u(x, r)\right]_{r=r_0} \text{ is } g_0(r_0)\cdot\frac{\Omega[xr_0]}{\Omega[x]}\cdot(\Phi^{(1)}[x, r_0])^k, \qquad (8')$$
$$(k=0, 1, 2, 3, \ldots \alpha-1)$$

as $g_0(r)$ does not vanish for the α first roots of the group in question. In like manner we obtain that for great absolute values of the variable x the dominant term of the solutions

$$u_{\alpha+k}(x)=\left[\frac{\partial^{\alpha+k}}{\partial r^{\alpha+k}}u(x, r)\right]_{r=r_\alpha} \text{ is } \binom{\alpha+k}{k}\cdot g_0^{(\alpha)}(r_\alpha)\cdot\frac{\Omega[xr_\alpha]}{\Omega[x]}\cdot(\Phi^{(1)}[x, r_\alpha])^k, \quad (8'')$$
$$(k=0, 1, 2, \ldots \beta-\alpha-1)$$

since $g_0(r)$ vanishes of order α for $r=r_\alpha$. In like manner for great absolute values of the variable x the dominant term of the solutions

$$u_{\beta+k}(x)=\left[\frac{\partial^{\beta+k}}{\partial r^{\beta+k}}u(x, r)\right]_{r=r_\beta} \text{ is } \binom{\beta+k}{k}\cdot g_0^{(\beta)}(r_\beta)\cdot\frac{\Omega[xr_\beta]}{\Omega[x]}\cdot(\Phi^{(1)}[x, r_\beta])^k, \quad (8''')$$
$$(k=0, 1, 2, \ldots \gamma-\beta-1)$$

since $g_0(r)$ vanishes of order β for $r=r_\beta$, and so forth. Finally we obtain that for great absolute values of the variable x the dominant term of the solutions

$$u_{\omega+k}(x)=\left[\frac{\partial^{\omega+k}}{\partial r^{\omega+k}}u(x, r)\right]_{r=r_\omega} \text{ is } \binom{\omega+k}{k}\cdot g_0^{(\omega)}(r_\omega)\cdot\frac{\Omega[xr_\omega]}{\Omega[x]}\cdot(\Phi^{(1)}[x, r_\omega])^k, \quad (8'''')$$
$$(k=0, 1, 2, \ldots \pi-\omega)$$

since $g_0(r)$ vanishes of order ω for $r=r_\omega$.

We exclude the case when two or more of the roots of the indicial equation are of equal absolute value without coinciding. Then we infer that it is always possible to arrange the solutions — according to decreasing absolute values of these roots and solutions corresponding to equal roots as has been done above — so that the quotient between a certain solution and the immediately following one tends to zero if x tends to infinity with the only restriction never to enter certain congruent circles. In this arrangement we denote the solutions

$$u^{(1)}(x),\ u^{(2)}(x),\ \ldots\ u^{(n)}(x).$$

Now we will make an important application of the obtained results, showing that our solutions are linearly independent in the sense that there cannot exist any relation of the following type among them

$$\sum_{i=1}^{i=n} k_i(x)\,.\,u^{(i)}(x)=0, \tag{9}$$

where $k_1(x), k_2(x), \ldots k_n(x)$ denote functions of multiplicative periodicity q, i. e., functions satisfying relations

$$k_i(xq)=k_i(x)\ \ (i=1, 2, 3, \ldots n), \tag{10}$$

(cf. Thomae (1), p. 259, Pincherle (2) and Rausenberger (1), p. 221), of which we suppose that they are uniform analytic functions, which do not all vanish identically. For assume the existence of such a relation and let $k_s(x)$ be the last of the functions $k_1(x), k_2(x), \ldots k_n(x)$ that does not vanish identically. Let $x=\alpha$ be a value for which $k_s(x)\neq 0$. We then divide the relation by $u_s(x)$ and perform a discontinuous limit-passage to infinity in letting the variable assume the successive congruent values $\alpha, \frac{\alpha}{q}, \frac{\alpha}{q^2}, \frac{\alpha}{q^3}, \ldots$, which always if α is chosen suitably are situated without the excluded congruent circles. As $\frac{u^{(i)}(x)}{u^{(i+1)}(x)}$ tends to zero by this limit-passage while $k_i(x)$ does not change, we must have $k_s(\alpha)=0$ contrary to our supposition. Hence any relation (9) cannot exist.

Such a system of linearly independent solutions we term a fundamental system of solutions. It may be emphasized that we have confined ourselves to analytic solutions by excluding non-analytic functions of multiplicative periodicity. (Cf. Nörlund (3).) It is obvious that, on the same condition, such a relation (9) cannot exist if the following determinant does not vanish identically

$$\operatorname{Det} u^{(i)}(xq^{k-1}). \tag{11}$$
$$(i, k=1, 2, 3, \ldots n)$$

Hence we have got another method to show that our solutions are linearly independent, which enables us to treat the case also that has been hitherto disregarded when two or more of the roots of the indicial equation are of equal absolute value without coinciding. Substituting the solution $u^{(i)}(x)$ in this determinant one finds that it may be written as a product of factors of the form $\frac{\Omega[xr]}{\Omega[x]}$ (and $\Phi^{(1)}[x, r]$ in case of equal roots) and a determinant, which tends to a finite limit $\neq 0$, when x tends to infinity. Then we conclude that the determinant (11) does not vanish identically and hence that no relation of type (9) can exist.

§ 4.

It has been shown in § 2 that the geometric difference equation (5) there has a solution of the form

$$\left.\begin{aligned} u(x) &= \frac{\Omega[xr]}{\Omega[x]} \cdot v(x), \\ \text{where} \qquad & \\ v(x) &= \sum_{\nu=0}^{\nu=\infty} \frac{g_\nu(r)}{(1-xr)\left(1-\frac{xr}{q}\right)\cdots\left(1-\frac{xr}{q^{\nu-1}}\right)} \end{aligned}\right\} \qquad (1)$$

converges for $|x| > \bar{R}$ at least. By an analytic continuation performed by aid of the difference equation itself we may obtain information about the analytic character of this function for $|x| \leqq \bar{R}$ also. Substituting the expression (1) for $u(x)$ in the difference equation (5) of § 2 transformed by aid of (3) of § 2 we obtain the following difference equation for $v(x)$

$$v(x) = \sum_{h=1}^{h=n} \frac{P_h(x)}{(1-xr)\left(1-\frac{xr}{q}\right)\cdots\left(1-\frac{xr}{q^{h-1}}\right)} \cdot v\left(\frac{x}{q^h}\right), \qquad (2)$$

where $P_h(x)$ is a linear combination of the functions $\bar{G}_i\left(\frac{x}{q^n}\right)(i = n-h, n-h+1, \ldots n)$ with coefficients that are polynomials in x, which do not depend on r. (It may be remembered that we have supposed $\bar{G}_n(x) = 1$, compare pp.

11 and 12.) The functions $\bar{G}_i(x)$ are by hypothesis holomorphic in the neighbourhood of infinity. Now we suppose, moreover, that they admit of an analytic continuation to every point of the x-plane, perhaps with the exception of the origin, and that their only singularities are at the following points

$$\beta_1, \beta_2, \beta_3, \ldots \beta_\nu \ldots, \tag{3}$$

which may occur in finite or infinite number. By substituting in (2) instead of x successively $\frac{x}{q}, \frac{x}{q^2}, \ldots \frac{x}{q^\mu}$ we obtain a system of equations, out of which we may eliminate $v\left(\frac{x}{q}\right), v\left(\frac{x}{q^2}\right), \ldots v\left(\frac{x}{q^\mu}\right)$, which gives the following relation

$$v(x) = \sum_{s=1}^{s=n} \frac{R_s^{(\mu)}(x)}{(1-xr)\left(1-\frac{xr}{q}\right)\cdots\left(1-\frac{xr}{q^{\mu+s-1}}\right)} \cdot v\left(\frac{x}{q^{\mu+s}}\right), \tag{4}$$

where $R_s^{(\mu)}(x)$ denotes a sum of products each factor being of the form

$$P_i\left(\frac{x}{q^k}\right) \left(\begin{matrix} i = 1, 2, 3, \ldots n \\ k = 0, 1, 2, \ldots \mu \end{matrix}\right).$$

Hence the functions $R_s^{(\mu)}(x)$ cannot have singularities elsewhere than at the points

$$\beta_i \,.\, q^{\mu+h} \left(\begin{matrix} i = 1, 2, 3, \ldots \\ h = 0, 1, 2, \ldots \end{matrix}\right).$$

By aid of (4) we then conclude that $v(x)$ is an analytic function of x, which cannot have singularities elsewhere than at points belonging to the following sets

$$\left.\begin{matrix} \beta_i \,.\, q^s \left(\begin{matrix} i = 1, 2, 3, \ldots \\ s = n, n+1, n+2, \ldots \end{matrix}\right) \\ \text{and} \\ r^{-1} \,.\, q^\nu \;\; (\nu = 0, 1, 2, \ldots). \end{matrix}\right\} \tag{5}$$

These latter points are, if they do not coincide with points of the former kind and if they really are singular points, simple poles. (Compare the equation (3) of § 5.) That they are not always singular points may be seen from the particular case when the coefficients of the difference equation are constants, in which case the function $v(x)$ itself is reduced to a constant. (Compare p. 36.)

As to our solutions

$$u(x) = \frac{\Omega[xr]}{\Omega[x]} \cdot v(x)$$

we must add to the singular points (5) of $v(x)$ the poles $r^{-1} . q^{-s}$ $(s = 1, 2, 3, \ldots)$ of the function $\frac{\Omega[xr]}{\Omega[x]}$, which are simple poles at least if s is chosen sufficiently great and r is not equal to or congruent to 1. For then we have $v(r^{-1} . q^{-s}) \neq 0$, as for great absolute values of x $v(x)$ tends to $g_0(r)$, which quantity does not vanish.

The other solutions belonging to the same group are obtained as we have shown from the solution $u(x)$ by derivation with respect to r. Hence we conclude by aid of (18) of § 2 and the relation (4) above that none of these solutions can have singularities elsewhere than at points out of the following sets

$$\beta_i . q^s \begin{pmatrix} i = 1, 2, 3, \ldots \\ s = n, n+1, n+2, \ldots \end{pmatrix}$$

and

$$r^{-1} . q^{\nu} \, (\nu = 0, \pm 1, \pm 2, \ldots).$$

These latter points are, however, generally poles of higher order than the first.

Of what has been mentioned we infer that our solutions are uniform analytic functions of the variable x if this is the case with the coefficients $G_i(x)$ $(i = 0, 1, 2, \ldots n)$. This property of the solutions seems to be rather remarkable, especially as the functional relation in question is finite. (Compare Leau (1) p. 42.) In this an advantage of our method over those which have been employed earlier on the same problem by Grévy (1) and Carmichael (1) seems to lie. We will now, however, indicate how to pass from our solutions to theirs. We have seen that our solutions may have two kinds of singular points: 1) such as are situated on one-sided equiangular spirals traced through the points β towards the origin, 2) such as are situated on double-sided equiangular spirals through the points r^{-1}. Our solutions are, however, only determined save as to factors consisting of functions of multiplicative periodicity. Such a function we denote $k(x)$. Then we have

$$Q^i u\,(x) k(x) = k(x)\,.\,Q^i u(x). \tag{6}$$

Now we choose for $k(x)$ the following function (compare the analogous function by Thomae, *loc. cit.* (1), p. 262 and Smith, *loc. cit.* (1), p. 13)

$$k(x) = x^{-\alpha} \cdot \frac{\Pi[xr]}{\Pi[xrq^{\alpha}]}, \tag{7}$$

where α is an arbitrary real or imaginary quantity, while $\Pi[x]$ denotes the »factorielle réciproque» of Cauchy (2)

$$\Pi[x] = (1-x)(1-xq)(1-xq^2)\ldots\left(1-\frac{q}{x}\right)\left(1-\frac{q^2}{x}\right)\left(1-\frac{q^3}{x}\right)\ldots.$$

Then out of (1) we obtain a new solution of the form

$$\bar{u}(x) = u(x)\,.\,k(x) = x^{-\alpha} \cdot \frac{\Pi[xr]}{\Pi[xrq^{\alpha}]} \cdot \frac{\Omega[xr]}{\Omega[x]} \cdot v(x) = (xq)^{-\alpha}\,\frac{\Omega\left[\dfrac{1}{xrq^{\alpha+1}}\right]}{\Omega\left[\dfrac{1}{xrq}\right]} \cdot \frac{\Omega[xrq^{\alpha}]}{\Omega[x]} \cdot v(x).$$

Hence it is obvious that one may by choosing the quantity α suitably replace the generally double-sided pole spiral through the point r^{-1} by any other similar spiral. But if we choose α so that $rq^{\alpha} = 1$, i. e.,

$$\alpha = -\frac{\log r}{\log q}, \tag{8}$$

we obtain

$$\bar{u}(x) = r\,.\,x^{\frac{\log r}{\log q}} \cdot \frac{\left(1-\dfrac{1}{xr}\right)\left(1-\dfrac{q}{xr}\right)\left(1-\dfrac{q^2}{xr}\right)\ldots}{\left(1-\dfrac{1}{x}\right)\left(1-\dfrac{q}{x}\right)\left(1-\dfrac{q^2}{x}\right)\ldots} \cdot v(x). \tag{9}$$

In this and generally only in this case the double-sided pole spiral has been replaced by a one-sided spiral, running through the point $x = 1$ towards the origin and we obtain a solution of the following form

$$\bar{u}(x) = x^{\mu}\,.\,\bar{v}(x), \tag{9'}$$

where $\bar{v}(x)$ is holomorphic in the neighbourhood of infinity. This solution is identical with that obtained by Grévy and Carmichael, since, as may be easily shown, our indicial equation is identical with their characteristic equation.

Moreover, our method enables us to treat the case when the indicial equation has equal roots or roots whose quotient is a power of q with integer exponent, a case that Carmichael does not treat. According to (6) we have, if we still denote the left-hand side of the given geometric difference equation $P(\bar{u}(x))$,

$$P(\bar{u}(x)) = k(x) \,.\, P(u(x)),$$

i. e., on account of (7) and (8)

$$P(\bar{u}(x)) = x^{\frac{\log r}{\log q}} \cdot \frac{\Pi[xr]}{\Pi[x]} \cdot P(u(x))$$

and hence according to (17) of § 2

$$P(\bar{u}(x)) = x^{\frac{\log r}{\log q}} \cdot \frac{\Pi[xr]}{\Pi[x]} \cdot \frac{\Omega[xr]}{\Omega[x]} \cdot g_0(r) \,.\, \varphi_0(r).$$

By reasoning in the same manner as before we obtain a system of solutions analogous to the system (20) of § 2 the general form being

$$\bar{u}_k^{(h)}(x) = \left[\frac{\partial^h}{\partial r^h}\bar{u}(x)\right]_{r=r_k^{(h)}} = x^{\frac{\log r_k^{(h)}}{\log q}} \,.\, [\varphi_0(x) + \varphi_1(x) \,.\, \log x + \\ + \varphi_2(x) \,.\, (\log x)^2 + \ldots + \varphi_h(x) \,.\, (\log x)^h], \tag{10}$$

where $\varphi_0(x), \varphi_1(x), \ldots \varphi_h(x)$ denote functions holomorphic in the neighbourhood of infinity. This result is included in Grévy's general theorems.

As the function (7) generally is a many-valued function of x this is also the case with the functions $\bar{u}(x)$. Hence they may be regarded as solutions of the difference equation in question only in the sense that they satisfy this equation if suitable determinations are given to $\bar{u}(x)$, $\bar{u}(xq)$, $\ldots$ $\bar{u}(xq^n)$.

We may now sum up our results in the following

THEOREM: *A linear homogeneous geometric difference equation of the following type*

$$\sum_{h=0}^{h=n} \bar{G}_h(x) \,.\, (1-xq)(1-xq^2)\ldots(1-xq^h) \,.\, (q-1)^h \,.\, q^{-\frac{1}{2}h(h+1)} \,.\, Q^h u(x) = 0 \quad \text{(A)}$$

may be given, where the coefficients $\bar{G}_h(x)$ *are supposed to be holomorphic in the neighbourhood of infinity and thus admit of a representation in form of geometric factorial series of the following type*

$$\bar{G}_h(x) = \bar{a}_0^{(h)} + \sum_{s=0}^{s=\infty} \frac{\bar{a}_{s+1}^{(h)}}{(1-x)\left(1-\frac{x}{q}\right)\dots\left(1-\frac{x}{q^s}\right)}, \tag{B}$$

valid for $|x| > R_h$. *In particular we suppose* $\bar{G}_n(x) = 1$. *This equation admits of a system of* n *solutions, the general form being*

$$u(x) = \frac{\partial^k}{\partial r^k} \frac{\Omega[xr]}{\Omega[x]} v(x, r), \tag{C}$$

where $v(x, r)$ *as well as its partial derivatives with respect to* r *are analytic functions of* x, *holomorphic in the neighbourhood of infinity, while* $\Omega[x]$ *denotes the Heinean function*

$$\Omega[x] = \frac{(1-q)(1-q^2)(1-q^3)\dots}{(1-xq)(1-xq^2)(1-xq^3)\dots}.$$

r *is determined as a root of the following algebraical equation*

$$\sum_{i=0}^{i=n} (1-r)\left(1-\frac{r}{q}\right)\dots\left(1-\frac{r}{q^{i-1}}\right) \cdot \bar{a}_0^{(i)} = 0, \tag{D}$$

whereat it is supposed that $\sum_{i=0}^{i=n} \bar{a}_0^{(i)} \neq 0$.

This system of solutions constitutes a fundamental system of solutions in the sense that there cannot exist any linear relation among them with coefficients that are functions of multiplicative periodicity q, *i. e., functions satisfying relations* $k(xq) = k(x)$, *which are analytic and do not all vanish identically.*

The solutions cannot have singularities elsewhere than at points out of the following sets

$$\left.\begin{array}{l} \beta_i \cdot q^s \begin{pmatrix} i = 1, 2, 3, \dots \\ s = n, n+1, n+2, \dots \end{pmatrix} \\ \text{and} \\ r^{-1} \cdot q^{\nu} \quad (\nu = 0, \pm 1, \pm 2, \dots). \end{array}\right\} \tag{E}$$

Here $\beta_i\,(i=1,2,3,\ldots)$ *denotes the set of points that is constituted of the singular points of the coefficients* $\bar{G}_0(x), \bar{G}_1(x), \ldots \bar{G}_{n-1}(x)$.

If the coefficients $\bar{G}_0(x), \bar{G}_1(x), \ldots \bar{G}_{n-1}(x)$ *are uniform functions of the variable* x, *this is also the case with the solutions.*

§ 5.

In order to illustrate our method we will now treat some particular geometric difference equations of simple character. We have seen that all the solutions contain factors

$$\frac{\Omega[xr]}{\Omega[x]}=\frac{(1-xq)(1-xq^2)(1-xq^3)\ldots}{(1-xrq)(1-xrq^2)(1-xrq^3)\ldots}.$$

On account of simplicity we do not here consider the cases when the indicial equation has equal roots or roots whose quotient is a power of q with integer exponent. Of particular interest is the case when $\Omega[xr]:\Omega[x]$ reduces to a rational function of x, which happens when $r=q^s$, where s is a positive or negative integer or zero, and only then. We distinguish the cases $s=0, s<0$ and $s>0$. 1) $s=0, r=1$. A necessary and sufficient condition for this is that $\bar{a}_0^{(0)}=0$, according to the indicial equation (13) of § 2. In this case the factor $\Omega[xr]:\Omega]x]$ disappears and the corresponding solution is obtained in the form of a pure geometric factorial series. Hence it is holomorphic in the neighbourhood of infinity. (Cf. Grévy, *loc. cit.* (1), Chap. I.) 2) $s<0$. Then we have

$$\frac{\Omega[xr]}{\Omega[x]}=\frac{1}{(1-x)\left(1-\frac{x}{q}\right)\ldots\left(1-\frac{x}{q^{-s-1}}\right)},$$

and the corresponding solution is holomorphic in the neighbourhood of infinity. 3) $s>0$. Then

$$\frac{\Omega[xr]}{\Omega[x]}=(1-xq)(1-xq^2)\ldots(1-xq^s),$$

and the corresponding solution is the sum of two functions: a polynomial in x and a function of x that is holomorphic in the neighbourhood of infinity. Occasionally this latter function may disappear. This happens, for instance, in case of geometric difference equations of the following type

$$\sum_{i=0}^{i=n}(1-xq)(1-xq^2)\dots(1-xq^i)\,.\,a_i\,.\,Q^i u(x)=0, \tag{1}$$

i. e., equations with constant coefficients. While the indicial equation, which depends only on the constant terms of the coefficients, here is of the general form (13) of § 2, the system of equations (12) of § 2 assumes the form

$$g_i\,.\,\varphi_0\left(\frac{r}{q^i}\right)=0 \quad (i=0, 1, 2, \dots).$$

As we suppose that the indicial equation has no equal roots or roots whose quotient is a power of q with integer exponent we have $\varphi_0\left(\frac{r}{q^i}\right) \neq 0$ for every $i>0$. Hence g_i must vanish for every $i>0$, and our solutions are reduced to quotients between two Ω-functions

$$\frac{\Omega[xr_i]}{\Omega[x]}\ (i=1, 2, 3, \dots n),$$

where $r_1, r_2, \dots r_n$ denote the roots of the indicial equation. Here as in the sequel we do not consider factors that are constants or functions of multiplicative periodicity q. Thus we have obtained n solutions, which are holomorphic in the neighbourhood of the origin. If we further suppose that $a_0=a_1=\dots=a_{s-1}=0$, while $a_s \neq 0$, in which case there is a group of s roots, it is obvious that the difference equation

$$\sum_{i=s}^{i=n}(1-xq)(1-xq^2)\dots(1-xq^i)\,.\,a_i\,.\,Q^i u(x)=0$$

has the following rational solutions

$$1, (1-xq), (1-xq)(1-xq^2), \dots (1-xq)(1-xq^2)\dots(1-xq^{s-1}).$$

In particular, the difference equation $Q^n u(x)=0$ has a fundamental system of rational solutions, viz., the successive geometric factorials

$$1, (1-xq), (1-xq)(1-xq^2), \dots (1-xq)(1-xq^2)\dots(1-xq^{n-1}).$$

As another example we choose the case when the series (5′) of § 2 contain only the first two terms, i. e., the equation

$$\sum_{i=0}^{i=n}(1-xq)(1-xq^2)\ldots(1-xq^i)\cdot\left(\bar{a}_i+\frac{\bar{b}_i}{1-x}\right)\cdot(q-1)^i\cdot$$
$$\cdot\, q^{-\frac{1}{2}i(i+1)}\cdot Q^i u(x)=0. \tag{2}$$

(Cf. Nörlund, *loc. cit.*, (1), p. 213.)
We suppose $\bar{a}_n=1$ and $\bar{b}_n=0$. Hence we have

$$\varphi\left(x,\frac{r}{q^\nu}\right)=\varphi_0\left(\frac{r}{q^\nu}\right)+\frac{\psi\left(\frac{r}{q^\nu}\right)}{1-x},$$

where

$$\varphi_0\left(\frac{r}{q^\nu}\right)=\sum_{i=0}^{i=n}\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\ldots\left(1-\frac{r}{q^{\nu+i-1}}\right)\cdot\bar{a}_i$$

and

$$\psi\left(\frac{r}{q^\nu}\right)=\sum_{i=0}^{i=n-1}\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\ldots\left(1-\frac{r}{q^{\nu+i-1}}\right)\cdot\bar{b}_i.$$

By aid of the following development, valid for $|x|>1$, (compare (7) of § 1)

$$\frac{1}{1-x}=\frac{r}{q^\nu}\cdot\sum_{s=0}^{s=\infty}\frac{1}{q^s}\cdot\frac{\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\ldots\left(1-\frac{r}{q^{\nu+s-1}}\right)}{\left(1-\frac{xr}{q^\nu}\right)\left(1-\frac{xr}{q^{\nu+1}}\right)\ldots\left(1-\frac{xr}{q^{\nu+s}}\right)}$$

we then obtain

$$\varphi_{s+1}\left(\frac{r}{q^\nu}\right)=\frac{r}{q^\nu}\cdot\frac{\psi\left(\frac{r}{q^\nu}\right)}{q^s}\cdot\left(1-\frac{r}{q^\nu}\right)\left(1-\frac{r}{q^{\nu+1}}\right)\ldots\left(1-\frac{r}{q^{\nu+s-1}}\right)\quad(s=0,1,2,\ldots).$$

Hence we have

$$g_1(r)=-g_0(r)\cdot\frac{\varphi_1(r)}{\varphi_0\left(\frac{r}{q}\right)}$$

and for $\nu>1$

$$g_\nu(r) = g_1(r) \cdot \prod_{\lambda=2}^{\lambda=\nu} \Phi_0(rq^{-\lambda}) \Big/ \prod_{\lambda=2}^{\lambda=\nu} \varphi_0(rq^{-\lambda}),$$

where

$$\Phi_0(r) = \varphi_0(rq) \cdot \left(\frac{1}{q} - rq\right) - \varphi_1(rq)$$

is a polynomial in r of degree $n+1$. Let $r_1, r_2, \ldots r_n$ be the roots of $\varphi_0(r) = 0$ and $\varrho_0, \varrho_1, \ldots \varrho_n$ the roots of $\Phi_0(r) = 0$. Then

$$\frac{\Phi_0(r)}{\varphi_0(r)} = \frac{1}{q} \cdot \frac{\left(1 - \frac{r}{\varrho_0}\right)\left(1 - \frac{r}{\varrho_1}\right) \cdots \left(1 - \frac{r}{\varrho_n}\right)}{\left(1 - \frac{r}{r_1}\right)\left(1 - \frac{r}{r_2}\right) \cdots \left(1 - \frac{r}{r_n}\right)}.$$

Denote $a_0 = \frac{r}{\varrho_0} \cdot q^{-2}$, $a_1 = \frac{r}{\varrho_1} \cdot q^{-2}, \ldots a_n = \frac{r}{\varrho_n} \cdot q^{-2}$; $b_1 = \frac{r}{r_1} \cdot q^{-2}$, $b_2 = \frac{r}{r_2} \cdot q^{-2}$, $\ldots b_n = \frac{r}{r_n} \cdot q^{-2}$; $\xi = rxq^{-1}$.

Hence we obtain n solutions of the difference equation (2) by substituting for r successively $r_1, r_2, \ldots r_n$ in the following expression

$$u(x, r) = \frac{\Omega[xr]}{\Omega[x]} \cdot \left\{ g_0(r) + g_1(r) \cdot \frac{y(x, r)}{1 - xr} \right\},$$

where

$$y(x, r) = 1 + \sum_{\nu=1}^{\nu=\infty} \frac{1 - a_0}{1 - \xi} \cdot \frac{1 - \frac{a_0}{q}}{1 - \frac{\xi}{q}} \cdots \frac{1 - \frac{a_0}{q^{\nu-1}}}{1 - \frac{\xi}{q^{\nu-1}}} \cdot \frac{1 - a_1}{1 - b_1} \cdot \frac{1 - \frac{a_1}{q}}{1 - \frac{b_1}{q}} \cdots \frac{1 - \frac{a_1}{q^{\nu-1}}}{1 - \frac{b_1}{q^{\nu-1}}} \cdots$$

$$\cdots \frac{1 - a_n}{1 - b_n} \cdot \frac{1 - \frac{a_n}{q}}{1 - \frac{b_n}{q}} \cdots \frac{1 - \frac{a_n}{q^{\nu-1}}}{1 - \frac{b_n}{q^{\nu-1}}} \cdot q^{-\nu}$$

is a series of the type that Thomae (2) termed Heine-series of higher order. (Cf. Smith (1).) There exists, however, between our investigation and Thomae's the difference that the series above is a geometric factorial series in the variable x, while the corresponding series by Thomae is a power series in this variable.

As an interesting particular case suppose $r_1 = \varrho_1, r_2 = \varrho_2, \ldots r_n = \varrho_n$. Then $\Phi_0(r) = \left(q^{-1} - \frac{r}{\varrho_0} \cdot q^{-1}\right) \cdot \varphi_0(r)$. Hence $\varrho_0 = q^{-n-2}$ and thus $a_0 = rq^n$. In this case $y(x, r) : (1 - xr)$ reduces to the following series

$$\frac{y(x, r)}{1 - xr} = \frac{1}{1 - xr} + \sum_{\nu=1}^{\nu=\infty} \frac{1}{q^\nu} \cdot \frac{(1 - rq^n)(1 - rq^{n-1}) \ldots (1 - rq^{n-\nu+1})}{(1 - xr)\left(1 - \frac{xr}{q}\right) \ldots \left(1 - \frac{xr}{q^\nu}\right)},$$

which is the development of the function $r^{-1} : (q^n - x)$ in geometric factorial series (compare (7) of § 1). Hence

$$u(x) = \frac{\Omega[xr]}{\Omega[x]} \left\{ g_0(r) + \frac{g_1(r)}{r} \cdot \frac{1}{q^n - x} \right\}$$

and thus the only singular point of the solution $u(x)$ besides those of the function $\frac{\Omega[xr]}{\Omega[x]}$ is the point $x = q^n$, i. e., the first point of the sets (5) of § 4.

Now consider the following particular difference equation that is obtained out of the equation (A) p. 33 by supposing $n = 1$ and $\bar{G}_0(x) = (xq)^{-1}$

$$(1 - xq) \cdot (q - 1) \cdot q^{-1} \cdot Qu(x) + (xq)^{-1} \cdot u(x) = 0. \tag{3}$$

Here $r = 1$ and hence the corresponding solution $u(x)$ is holomorphic in the neighbourhood of infinity. One immediately finds that $\Omega\left[\frac{1}{xq}\right]$ is a solution of (3). Thus the solution $u(x)$ may differ from $\Omega\left[\frac{1}{xq}\right]$ only by a constant factor, as every uniform function of multiplicative periodicity that is holomorphic in the neighbourhood of infinity reduces to a constant. In this case the only singularities of the solution are, besides the essential singularity at the origin, the simple poles

$$x = q^\nu \quad (\nu = 0, 1, 2, \ldots).$$

Finally we mention the difference equation that is obtained out of (5) of § 2 by supposing

$$G_i(x) = \frac{(1 - xq^{i+1})(1 - xq^{i+2}) \dots (1 - xq^n)}{x^{n-i}} \cdot h_i(x) \quad (i = 0, 1, 2, \dots n),$$

where $h_i(x)$ denotes an entire function of x^{-1}, namely in the main the equation

$$\sum_{i=0}^{i=n} x^i \,.\, h_i(x) \,.\, Q^i u(x) = 0 \quad \text{or} \quad \sum_{i=0}^{i=n} \bar{h}_i(x) \,.\, u(xq^i) = 0,$$

where $\bar{h}_i(x)$ is an entire function of x^{-1}. This is an equation of the type treated by Mason (**1**).

Note.

We have defined p. 22 $\Phi^{(1)}[x, 1] = \sum_{\nu=1}^{\nu=\infty} \frac{xq^\nu}{1 - xq^\nu}$, where $|q| < 1$. Hence we have for $|x| < 1$ at least

$$\Phi^{(1)}[x, 1] = \sum_{\nu=1}^{\nu=\infty} \Big(\sum_{i=1}^{i=\infty} (xq^\nu)^i \Big) = \sum_{s=1}^{s=\infty} q^s \, . \, K_s(x),$$

where

$$K_s(x) = \sum_{k=1}^{k=s} \left\{\frac{s}{k}\right\} x^k.$$

The symbol $\left\{\frac{s}{k}\right\}$ we define in the following manner

$$\left\{\frac{s}{k}\right\} = \begin{cases} 1 \text{ if } s \equiv 0 \pmod{k} \\ 0 \text{ if } s \not\equiv 0 \pmod{k}. \end{cases}$$

Furthermore we have

$$(1 - q) \, . \, \Phi^{(1)}[x, 1] = \sum_{s=1}^{s=\infty} q^s \, . \, (K_s(x) - K_{s-1}(x)),$$

where $K_0(x) \equiv 0$. This development is valid for $|q| < 1$, while the series diverges for $q = 1$, since $\lim\limits_{s \to \infty} K_s(x)$ does not exist. It is however summable. For we have

$$s_n = \sum_{\nu=1}^{\nu=n} (K_\nu(x) - K_{\nu-1}(x)) = K_n(x)$$

and hence

$$s_n^{(1)} = \frac{s_1 + s_2 + \ldots + s_n}{n} = \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} K_\nu(x) = \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} \Big(\sum_{k=1}^{k=\nu} \left\{\frac{\nu}{k}\right\} x^k \Big) = \frac{1}{n} \cdot \sum_{k=1}^{k=n} x^k \cdot \Big(\sum_{\nu=k}^{\nu=n} \left\{\frac{\nu}{k}\right\} \Big).$$

Now we have $\sum_{\nu=k}^{\nu=n} \left\{\frac{\nu}{k}\right\} = \left[\frac{n}{k}\right]$, where the latter symbol denotes the greatest integer contained in $\frac{n}{k}$. Hence

$$s_n^{(1)} = \frac{1}{n} \cdot \sum_{k=1}^{k=n} x^k \cdot \left[\frac{n}{k}\right] = \sum_{k=1}^{k=n} \frac{x^k}{k} \cdot \frac{\left[\frac{n}{k}\right]}{\frac{n}{k}}.$$

But

$$\frac{n}{k} = \left[\frac{n}{k}\right] + \frac{m}{k},$$

where m denotes a positive integer or zero, which depends upon n and k but which is always less than k. Hence we have

$$s_n^{(1)} = \sum_{k=1}^{k=n} \frac{x^k}{k}\left(1 - \frac{m}{n}\right) = \sum_{k=1}^{k=n} \frac{x^k}{k} - \frac{1}{n} \cdot \sum_{k=1}^{k=n} \frac{m}{k} \cdot x^k.$$

As $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{k=n} \frac{m}{k} \cdot x^k = 0$, if $|x| < 1$, while $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k=n} \frac{x^k}{k} = -\log(1-x)$, we obtain $\lim\limits_{n\to\infty} s_n^{(1)} = -\log(1-x)$. Thus it follows, according to the extension of the well-known Abelian theorem on the continuity of power series, that has been given by Frobenius (3), that, at least for real x such that $|x| < 1$ and for real and increasing q,

$$\lim_{q\to 1} (q-1) \,.\, \Phi^{(1)}[x, 1] = \log(1-x).$$

(Cf. Hardy, *Proc. London Math. Soc.*, (Ser. 2) Vol. 4, p. 247 (1907).)

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